METU - NCC

DIFFERENTIAL EQUATIONS MIDTERM 1		
Code : *MAT 219* Acad.Year : *2014-2015* Semester : *FALL* Date : *02.11.2014* Time : *9:40* Duration : *120 min*	Last Name: Name : Student # : SOLUTIONS Signature :	List #:
	6 QUESTIONS ON 6 PAGES TOTAL 100 POINTS	

1. (18)	2. (14)	3. (22)	4. (16)	5. (18)	6. (12)		

Please draw a $\boxed{\text{box}}$ around your answers. No calculators, cell-phones, notes, etc. allowed.

1. *(9+9pts)* The parts below are about the differential equation $y' = \dfrac{x^2}{y(1+x^3)}$.

(A) Solve the differential equation.

This is a separable equation:

$$yy' = \frac{x^2}{1+x^3}$$

By direct integration, we get

$$\frac{1}{2}y^2 = \frac{1}{3}\ln|1+x^3| + c$$

$$y = \pm\sqrt{\frac{2}{3}\ln|1+x^3| + c}$$

(B) For the same differential equation, write an initial condition whose solution satisfies the property given in each part below. **Explain your answers.**

(i) Unique solution.

We use the existence and uniqueness theorem for nonlinear diff. eqn $y' = f(x,y)$:

$$f(x,y) = \frac{x^2}{y(1+x^3)} = \frac{x^2}{y(1+x)(1-x+x^2)} \Rightarrow \text{discontinuous along } y=0 \text{ or } x=-1$$

$$\frac{\partial f}{\partial y} = -\frac{1}{y^2}\frac{x^2}{1+x^3} = -\frac{1}{y^2}\frac{x^2}{(1+x)(1-x+x^2)}$$

Unique solution for $y(a)=b$ if $a \neq -1$ or $b=0$ e.g. $y(2)=5$.

(ii) No solution.

No solution for

$y(-1) =$ ANY VALUE.

(iii) More than one solution. if $y(a)=0$ for $a \neq -1$

such as $y(1)=0$; $y = \pm\sqrt{\frac{2}{3}\ln|1+x^3| - \frac{2}{3}\ln 2}$

2. *(10+4pts)* This problem has two unrelated parts.

(A) Solve the initial value problem. $\quad t^3\, y' + 4t^2\, y = e^{-t} \quad$ with $\quad y(-1) = 1.$

Multiply the diff eqn by t or by finding integrating factor $\mu(t)$:

$$t^4 y' + 4t^3 y = te^{-t}$$

$$(t^4 y)' = te^{-t} \Rightarrow t^4 y = -te^{-t} - e^{-t} + c$$

$$\Rightarrow y = -\frac{1}{t^3}e^{-t} - \frac{1}{t^4}e^{-t} + \frac{c}{t^4}$$

$$y(-1) = \underbrace{-\frac{1}{(-1)}e^{1} - \frac{1}{1}e^{1}}_{0} + \frac{c}{1}$$

$$1 = 0 + c \Rightarrow c = 1$$

$$\Rightarrow \boxed{y(t) = -\frac{1}{t^3}e^{-t} - \frac{1}{t^4}e^{-t} + \frac{1}{t^4}}$$

OR

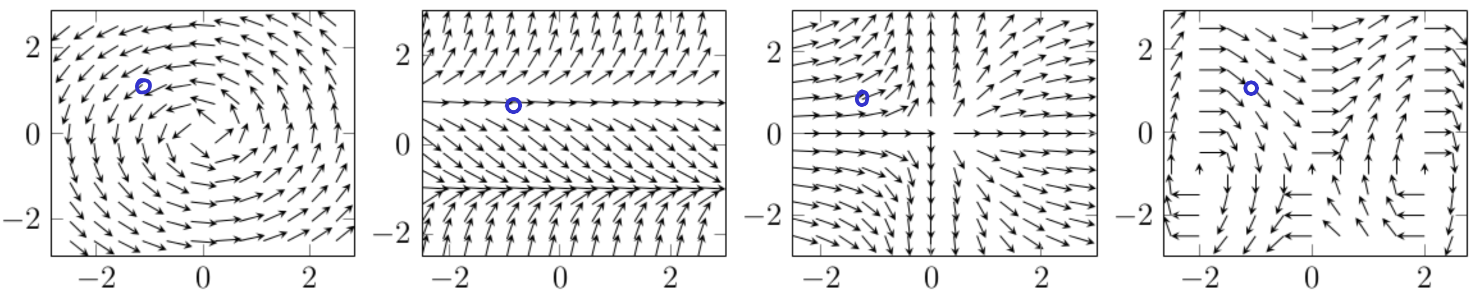
Integrating factor can be found by:

$$y' + \frac{4}{t}y = \frac{e^{-t}}{t^3} \Rightarrow \mu(t) = e^{\int \frac{4}{t}dt} = e^{4\ln t} = t^4$$

Multiply by $\mu(t) = t^4$:

$$t^4 y' + 4t^3 y = te^{-4}$$

(B) Match the following direction fields with their differential equations.



(I) (II) (III) (IV)

- $\boxed{\text{II}}$ $\quad y' + 1 - y^2 = 0 \Rightarrow y' = y^2 - 1$
  $y'|_{(-1,1)} = 1^2 - 1 = 0$
- $\boxed{\text{I}}$ $\quad y\,y' + x = 0 \Rightarrow y' = -\frac{x}{y}$
  $y'|_{(-1,1)} = -\frac{(-1)}{1} = 1.$
- $\boxed{\text{III}}$ $\quad x^2\, y' - y = 0$
  $y' = \frac{y}{x^2} \xRightarrow{\text{at }(-1,1)} y' = \frac{1}{1} = 1$
- $\boxed{\text{IV}}$ $\quad (y-1)\, y' + x^3 - 4x = 0$
  $y' = \frac{4x - x^3}{y-1}$ at $(-1,1)$ $y' = \frac{3}{-2}$

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3. *(10+12)* Given the differential equation $\dfrac{dy}{dx} = \dfrac{-xy - y^2}{2x^2 + 5xy}$

(A) Solve it (as a homogeneous DE) by using the substitution $v = \dfrac{y}{x}$ so that it becomes separable.

$$\frac{dy}{dx} = \frac{\frac{-xy-y^2}{x^2}}{\frac{2x^2+5xy}{x^2}} = \frac{-\frac{y}{x} - \left(\frac{y}{x}\right)^2}{2+5\frac{y}{x}} \xrightarrow[v'x+v=y']{\text{substitute } v=\frac{y}{x}} xv'+v = \frac{-v-v^2}{2+5v}$$

$$xv' = \frac{-v-v^2}{2+5v} - v = \frac{-v-v^2-2v-5v^2}{2+5v} = \frac{-3v-6v^2}{2+5v}$$

$$-\frac{1}{3}\,\frac{2+5v}{v+2v^2}\,dv = \frac{dx}{x} \quad \left(\text{note: } \frac{2+5v}{v+2v^2} = \frac{2+5v}{v(1+2v)} = \frac{2}{v} + \frac{1}{1+2v}\right)$$

$$-\frac{1}{3}\left(\frac{2}{v} + \frac{1}{1+2v}\right)dv = \frac{dx}{x}$$

$$-\frac{1}{3}\left(2\ln|v| + \frac{1}{2}\ln|1+2v|\right) = \ln|x| + c$$

$$c = +4\ln|v| + \frac{1}{2}\ln|1+2v| + 6\ln|x|$$

$$c = x^6\, v^4\,(1+2v) \qquad v = \frac{y}{x}$$

$$c = x^6\left(\frac{y}{x}\right)^4\left(1+2\frac{y}{x}\right)$$

$$\Rightarrow \boxed{c = x^2y^4\left(1+2\frac{y}{x}\right) = x^2y^4 + 2xy^5}$$

(B) Solve the same differential equation by finding an integrating factor so that it becomes exact.

$$(xy+y^2)\,dx + (2x^2+5xy)\,dy = 0$$

* Is it exact? $\dfrac{\partial}{\partial y}(xy+y^2) = x+2y \neq \dfrac{\partial}{\partial x}(2x^2+5xy) = 4x+5y$

* Try a factor $\mu(x)$:

$$\frac{\partial}{\partial y}\left((xy+y^2)\mu(x)\right) = (x+2y)\mu(x) = \frac{\partial}{\partial x}\left((2x^2+5xy)\mu(x)\right) = (4x+5y)\mu + (2x^2+5xy)\mu'$$

$$\Rightarrow (x+2y)\mu - (4x+5y)\mu = (2x^2+5xy)\mu' \Rightarrow (-3x-3y)\mu = x(2x+5y)\mu'$$

$$\frac{\mu'}{\mu} = \frac{-3x-3y}{2x+5y} \quad \times$$

* Try a factor $\mu(y)$:

$$\frac{\partial}{\partial y}\left((xy+y^2)\mu(y)\right) = (xy+y^2)\mu'(y) + (x+2y)\mu(y)$$

$$\frac{\partial}{\partial x}\left((2x^2+5xy)\mu(y)\right) = (4x+5y)\mu(y)$$

$$\Rightarrow (xy+y^2)\mu' = (4x+5y)\mu - (x+2y)\mu$$

$$y(x+y)\mu' = (3x+3y)\mu$$

$$\frac{\mu'}{\mu} = \frac{3(x+y)}{y(x+y)} = \frac{3}{y}$$

$$\therefore \boxed{\mu = y^3}$$

Use $\mu = y^3$ for integrating factor:

$$\Rightarrow (xy^4+y^5)\,dx + (2x^2y^3+5xy^4)\,dy = 0$$

$$\Rightarrow F = \frac{1}{2}x^2y^4 + xy^5 + g(y)$$

$$\Rightarrow \frac{\partial F}{\partial y} = 2x^2y^3 + 5xy^4 + g'(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = \text{const.}$$

$$\Rightarrow \boxed{\frac{1}{2}x^2y^4 + xy^5 = \text{constant}}$$

4. *(5+8+3)* A tank with a capacity of 500 L originally contains 200 L of water with 100 kg of salt in solution. Water containing 1 kg of salt per liter is entering at a rate of 3 L/min, and the mixture is allowed to leave the tank at a rate of 2 L/min.

(A) Write a differential equation for the change in the amount of salt in the tank before the time when the tank is full. Include initial values.

Volume of water in the tank $= 200 + 3t - 2t = 200 + t$ liters

$Q(t)$ = amount of salt in the Tank (kg)

$$\frac{dQ}{dt} = 3\ lt/min \cdot 1\ kg/lt - 2\ lt/min \cdot \frac{Q}{200+t}\ \frac{kg}{lt}\ ;\ Q(0) = 100$$

(B) Solve the differential equation and compute the amount of salt in the tank when it is full.

$$\frac{dQ}{dt} + \frac{2}{200+t} Q = 3$$

find integrating factor for linear eqn:

$$\mu(t) = e^{\int \frac{2}{200+t} dt} = (200+t)^2$$

Multiply by $\mu(t) = (200+t)^2$

$$(200+t)^2 \frac{dQ}{dt} + 2(200+t)Q = 3(200+t)^2$$

$$\frac{d}{dt}\left\{(200+t)^2 Q\right\} = 3(200+t)^2$$

Integrate

$$(200+t)^2 Q = (200+t)^3 + c$$

$$Q = (200+t) + \frac{c}{(200+t)^2}$$

$$100 = Q(0) = 200 + \frac{c}{40000}$$

$$\frac{c}{40000} = -100 \Rightarrow c = -4 \cdot 10^6$$

$$\boxed{Q(t) = (200+t) - 4 \cdot 10^6 / (200+t)^2}$$

Tank is full if

$200 + t = 500$

ie $t = 300$ mins.

$$Q(300) = (200+300) - \frac{4 \cdot 10^6}{(500)^2}$$

$$= 500 - \frac{4 \cdot 10^6}{25 \cdot 10^4}$$

$$= 500 - 4 \cdot \frac{10^2}{25}$$

$$Q(300) = 500 - 4 \cdot 4 = \underline{\underline{484}}$$

(C) After the tank is full, it will overflow, so that water leaves the tank at 3 L/min (the same rate as it enters). Write a new differential equation for the change in the amount of salt after the tank is full. Include initial values.

Volume $= 500$ L

$$\frac{dQ}{dt} = 3\ lt/min \cdot 1\ kg/lt - 3\ lt/min \cdot \frac{Q}{500}\ kg/lt$$

$$Q(0) = 484$$

5. *(10+8pts)* Calculate the eigenvalues and eigenvectors of the following matrices.

(A) $A = \begin{bmatrix} 4 & -6 \\ 6 & -9 \end{bmatrix}$.

Eigenvalue equation:

$\det(A - \lambda I) = 0$

$\parallel$

$\lambda^2 + 5\lambda \quad = 0$

Eigenvalues: $0, -5$

An Eigenvector for $\lambda = 0$: $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ All eigenvectors for $\underline{0}$: $\begin{bmatrix} 3 \\ 2 \end{bmatrix} c \quad c \in \mathbb{R}$

An Eigenvector for $\lambda = -5$: $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ All eigenvectors for $\underline{-5}$: $\begin{bmatrix} 2 \\ 3 \end{bmatrix} d \quad d \in \mathbb{R}$

(B) $B = \begin{bmatrix} 4 & -6 & 0 \\ 6 & -9 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

Eigenvalues: $0, \quad -5, \ 3$

Eigenvectors: $\begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

6. *(12pts)* Solve the following system of linear differential equations

$$\frac{dx_1}{dt} = x_2 \qquad \text{with} \qquad x_1(0) = 2$$
$$\frac{dx_2}{dt} = x_1 \qquad \qquad x_2(0) = -4.$$

Note that $\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ and $\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}$.

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ eigenvector with eigenvalue $+1$. $\qquad$ $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ eigenvector with eigenvalue (-1)

$$\left.\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= x_1 \end{aligned}\right\} \xRightarrow{x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \frac{dx}{dt} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x$$

ANY SOLUTION IS OF THE FORM $x = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{1\cdot t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{(-1)t}$

From $x(0) = \begin{bmatrix} 2 \\ -4 \end{bmatrix}$, we find $\begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} c_1 e^0 + c_2 e^{-0} \\ c_1 e^0 - c_2 e^{-0} \end{bmatrix}$

solve for c_1, c_2: $\quad c_1 = -1, \quad c_2 = 3$

$$\Rightarrow \quad x(t) = (-1)\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t + 3 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

ALTERNATIVELY ONE CAN USE A DIAGONALIZATION

OF $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{S} \underbrace{\begin{bmatrix} 1 & \\ & -1 \end{bmatrix}}_{\Lambda} \cdot S^{-1}$

Let $y = S^{-1}x$ then $\frac{dx}{dt} = Ax$ becomes $\frac{dy}{dt} = \Lambda y$

which is to say $\frac{dy_1}{dt} = y_1$ & $\frac{dy_2}{dt} = -y_2$

then $y_1 = c_1 e^t$ & $y_2 = c_2 e^{-t}$

Since $x = Sy = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} c_1 e^t \\ c_2 e^{-t} \end{bmatrix} = \begin{bmatrix} c_1 e^t + c_2 e^{-t} \\ c_1 e^t - c_2 e^{-t} \end{bmatrix}$.

FOR THE REST, SEE